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Logarithmic operators in $SL(2, \mathbb{R})$ WZNW model, singletons and $AdS_3/(L)CFT_2$ correspondence

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Abstract
We discuss the role of singletons and logarithmic operators in AdS_3 string theory in the context of AdS_3/CFT_2 correspondence.

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1. Introduction

String theory in AdS_3 background has been widely studied as an example of string theory in non-trivial curved spacetime background in last one decade and recently in the context of AdS/CFT correspondence [1, 2]. Worldsheet theory is described by the $SL(2, \mathbb{R})$ WZNW model.

Correlation functions in the WZNW model satisfy Knizhnik–Zamolodchikov (KZ) equation [3]. In [4–6] some solutions of Knizhnik–Zamolodchikov equation for four-point functions of $SL(2, \mathbb{R})$ primaries were found which were logarithmically singular as the two operators approach each other. It was suggested that it might be necessary to include contribution of logarithmic operators in operator product expansion (OPE) and $SL(2, \mathbb{R})$ WZNW conformal theory is a logarithmic conformal field theory (LCFT) [7, 8] (see also the recent reviews [9] and references therein). The solutions involved one or more operators in non-trivial identity representation. In addition to trivial identity which is just a constant, we also have a non-trivial identity in $SL(2, \mathbb{R})$ representation theory (see below). This corresponds to singletons. These are special finite-dimensional representations which lie at the limit of unitary bound [10].

A representation of $SL(2, \mathbb{R})$ group [11] element can be labelled by eigenvalue of quadratic Casimir, $-j(j+1)$ and eigenvalue, m of one of generators say J^3. We have (i) continuous representation, C_j, $j+\alpha \notin \mathbb{Z}, \alpha$ being the fractional part of m; (ii) discrete series of lowest weight type, $D_j^+, 2j \notin \mathbb{Z}^+ \cup 0$; (iii) discrete series of highest weight type, $D_j^-, 2j \notin \mathbb{Z}^+ \cup 0$ and (iv) finite-dimensional representation $I_j, 2j \in \mathbb{Z}^+ \cup 0$. We have suppressed the other label m and fractional part of m. Continuous series is unitary for

$j = -\frac{1}{2} + \mathrm{i}\rho$, $\rho \in \mathbb{R}$, and for an exceptional interval $-\frac{1}{2} < j < \frac{1}{2}$, discrete series D_j^- and D_j^+ for $j < 0$ and finite-dimensional representation I_j for $j = 0$ only.

Finite-dimensional representation can be embedded in a reducible but indecomposable representation with eigenvalues of J^3 unrestricted. Singletons are part of this non-unitary indecomposable finite-dimensional representation $j = 0$. It should be distinguished from trivial $j = 0$ representation which appears if we restrict ourselves to unitary irreducible representations.

Singleton representation is somewhat special. Two-point function of singleton modes in AdS_3 has two solutions, one of them is logarithmic (see, for example, [12] and references therein). One can discard the logarithmic solution by imposing vanishing flux condition at infinity in favour of the other. However, it can still give logarithmic singularities in four-point functions. It was shown in [13] that if one considers singletons in the bulk of AdS_3, then two-point functions are logarithmic in boundary conformal field theory (see also [14]). It was conjectured [13] that boundary conformal field theory is logarithmic conformal field theory. Calculation of absorption cross section for gauge bosons in the bulk of AdS_3 [15, 16] (further discussed in [17] in the context of AdS/LCFT correspondence) lends some support to this conjecture.

It was argued that $j = -\frac{1}{2} \in D_j^+$ representation determines whether boundary CFT is logarithmic or not [18]. $j = -\frac{1}{2}$ was considered as unitary bound in $SL(2,\mathbb{R})$ WZNW model in that reference. AdS/CFT correspondence allows us to define correlation function for $j > -\frac{1}{2}$ beyond the bound $j = -\frac{1}{2}$. We show that they are well behaved unless we reach the unitary bound $j = 0$.

At $j = 0$ we have to take into account the singleton representation and correlation functions can be logarithmic.

The paper is organized as follows. In section 2, we discuss the origin of logarithmic operators in AdS_3 string theory. In section 3, it is shown how singletons can give logarithmic correlation functions in boundary CFT. In section 4, we discuss correlation functions of fields with $j > -\frac{1}{2}$ and dimension-dependent normalization of the fields in the interval $0 \geqslant j \geqslant -\frac{1}{2}$. We conclude with brief discussion and summary.

2. Logarithmic operators in $SL(2,\mathbb{R})$ WZNW model

Primary fields of $SL(2,\mathbb{R})$ WZNW are labelled by the quantum numbers of global $SL(2,\mathbb{R})$ symmetry. Generators of global $SL(2,\mathbb{R})$ group satisfy

$$[J^+, J^-] = -2J^3 \qquad [J^3, J^\pm] = \pm J^\pm. \tag{1}$$

The quadratic Casimir is

$$C_2 = \eta_{ab} J^a J^b = \tfrac{1}{2}(J^+J^- + J^-J^+) - J^3J^3 = -j(j+1). \tag{2}$$

A representation for the $SL(2,\mathbb{R})$ generators is given by [19]

$$D^+ = -x^2\frac{\partial}{\partial x} + 2jx \qquad D^- = -\frac{\partial}{\partial x} \qquad D^3 = x\frac{\partial}{\partial x} - j. \tag{3}$$

Primary fields $\phi_j(x, z)$ satisfy

$$J^a \phi_j(x, z) = D^a(x)\phi_j(x, z) \tag{4}$$

where J^a are the zero modes of the $SL(2,\mathbb{R})$ current algebra and $D^a(x)$ is as given in equation (3). The fields $\phi_j(x, z)$ are also primary with respect to the Virasoro algebra with L_0 eigenvalue:

$$\Delta_j = -\frac{j(j+1)}{k-2}. \tag{5}$$

For the discussion of correlation function we consider Euclidean version of $SL(2,\mathbb{R})$, which is $SL(2,C)/SU(2)$. Primary fields of the Euclidean model are given by [20, 21]

$$\phi_j(x,z) \sim \left[(1,-x)g\begin{pmatrix}1\\ -\bar{x}\end{pmatrix}\right] = [(\gamma - x)(\bar{\gamma}-\bar{x})\,e^{2\phi} + e^{-2\phi}]^{2j} \tag{6}$$

where $g \in SL(2,C)$ and γ, $\bar{\gamma}$ and ϕ are $SL(2,C)$ coordinates.

The primary field $\phi_j(x,z)$ has the form of a bulk to boundary Green function for a scalar field in AdS_3 and $(x,\bar{x})$ has the interpretation of boundary coordinates [20].

Correlation functions in WZNW model satisfy a set of partial differential equations known as Knizhnik–Zamolodchikov (KZ) equation [3]. For two- and three-point functions it gives no new information. Four-point functions are determined by conformal invariance up to a function, $F(x,z)$, of conformally invariant cross ratios $z = \frac{z_{12}z_{34}}{z_{13}z_{24}}$, $x = \frac{x_{12}x_{34}}{x_{13}x_{24}}$. The Knizhnik–Zamolodchikov (KZ) equation for the four-point functions becomes a partial differential equation for conformal blocks $F(x,z)$.

Existence of logarithmic operators is signalled by the presence of logarithmic singularities of cross ratios in four-point solutions. In [5] we found some exact solutions with logarithms of cross ratios in conformal blocks.

Consider a solution of the KZ equation with logarithmic singularities of cross ratios $\left(j = j_1 = j_3 \neq -\frac{1}{2},\ j_2 = j_4 = 0\right)$:

$$\langle\phi_j(x_1,z_1)\phi_0(x_2,z_2)\phi_j(x_3,z_3)\phi_0(x_4,z_4)\rangle = |z_{13}|^{-2h}|x_{13}|^{-2j}\left[A\left(\ln\left|\frac{1-x}{x}\right| + \frac{2j+1}{k-2}\ln\left|\frac{1-z}{z}\right|\right) + B\right]. \tag{7}$$

Now taking the $1\to 2$ and $3\to 4$ limits which correspond to $x,z\to 0$, we can expand our solution as

$$\langle\phi_j(x_1,z_1)\phi_0(x_2,z_2)\phi_j(x_3,z_3)\phi_0(x_4,z_4)\rangle = |z_{13}|^{-2h}|x_{13}|^{-2j}\Big[\ln|x_{12}|\ln|x_{34}| - 2\ln|x_{24}| + \frac{2j+1}{k-2}(\ln|z_{12}| + \ln|z_{34}| - 2\ln|z_{24}|) + \cdots\Big] \tag{8}$$

where we use $\ln x = \ln x_{12} + \ln x_{34} - \ln x_{13} - \ln x_{24} = \ln x_{12} + \ln x_{34} - 2\ln x_{24} + \cdots$ and similarly for z. '$\cdots$' stands for subleading terms.

The above solution is consistent if the OPE is of the following form:

$$\begin{aligned}\phi_j(x_1,z_1)\phi_0(x_2,z_2) &= E_j^x(x_2,z_2)\ln|x_{12}| + E_j^z(x_2,z_2)\ln|z_{12}| + F_j(x_2,z_2) + \cdots\\ \phi_j(x_3,z_3)\phi_0(x_4,z_4) &= E_j^x(x_4,z_4)\ln|x_{34}| + E_j^z(x_4,z_4)\ln|z_{34}| + F_j(x_4,z_4) + \cdots.\end{aligned} \tag{9}$$

The two-point functions of the fields E_j^a and F_j are $\left(j \neq -\frac{1}{2}\right)$

$$\begin{aligned}&\left\langle E_j^a E_j^b\right\rangle = 0 \qquad \left\langle E_j^z(x,z)F_j(0,0)\right\rangle = \frac{2j+1}{(k-2)}|z|^{-2h}|x|^{-2j}\ \left\langle E_j^x(x,z)F_j(0,0)\right\rangle = |z|^{-2h}|x|^{-2j}\\ &\langle F_j(x,z)F_j(0,0)\rangle = -2|z|^{-2h}|x|^{-2j}\left(\frac{2j+1}{k-2}\ln|z| + \ln|x|\right).\end{aligned} \tag{10}$$

If $j = -\frac{1}{2}$ it is consistent at this level to set $E_j^z = 0$ and we have logarithmic blocks in isotopic space only.

Now let us see how logarithmic operators can occur in AdS_3 string theory. The fields E_j^a and F_j are degenerate and span the Jordan cell structure of the Lie algebra. This is the well-known story in logarithmic CFTs. One has to deal with reducible but indecomposable representation of the corresponding symmetry algebra. So one has to look for extension of $SL(2,\mathbb{R})$ modules to include the fields E_j^a and F_j.

The $SL(2,\mathbb{R})$ module is degenerate for the following values of j [22]:

$$j_{r,s} = \frac{r-1}{2} + \frac{s-1}{2}(k-2). \tag{11}$$

Null states exist for $r, s \in \mathcal{Z}$ and one can consistently extend chiral symmetry algebra to include the fields E_j^a and F_j satisfying the correlation functions (10) [23]. The extended module for $j_{1,1} \equiv 0$ is also known as singleton representation.

Normalizable wavefunctions in AdS_3 exist for $j < 0$ only and they are square integrable only for $j < -\frac{1}{2}$. So if one is dealing with square integrable wavefunctions, it is sufficient to restrict the fields with $j < -\frac{1}{2}$. However, to account for existence of logarithmic pairs of the fields E_j^a and F_j one has to deal with representation to include fields with $j \leqslant 0$. At $j = 0$, one encounters singleton representation. It plays a special role in bulk-boundary correspondence as discussed in the next section.

3. Singletons and logarithmic operators in boundary CFT

Anti-de Sitter group $SO(d,2)$ in $d+1$ spacetime dimensions has special representations called singletons. They are special because they saturate the unitarity bound and have singular flat spacetime limit as discovered by Dirac [24]. They can also be considered as topological fields living on the boundary of AdS spacetime [25, 26]. Since the group $SO(d,2)$ is the conformal symmetry group in d dimensions, we can consider them the representations of conformal group in d dimensions. Singletons are in the indecomposable representation of conformal algebra and give logarithmic correlation functions on the boundary of AdS_d [13, 14].

Theory of singletons can be formulated in terms of Flato–Fronsdal dipole [27] field which satisfies the following equations of motion:

$$(\nabla + m^2)A + B = 0 \qquad (\nabla + m^2)B = 0. \tag{12}$$

These can be derived from the following form of the action in AdS_{d+1},

$$S = \int \mathrm{d}^{d+1}x\sqrt{g}\left(g^{\mu\nu}\partial_\mu A \partial_\nu B - m^2 AB - \frac{\mu^2}{2}B^2\right) \tag{13}$$

with $m^2 = \Delta(\Delta - d)$ and $\mu^2 = (2\Delta - d)/2$. Singleton corresponds to $\Delta = \Delta_0 = \frac{d-2}{2} \Rightarrow \mu^2 = -1$. For AdS_3, $d = 2$ and $\Delta = \Delta_0 = 0 = m^2$, so we shall consider the limit $\Delta \to 0$.

The above form of the action is shown to give logarithmic two-point functions on the boundary [14] of AdS spacetime. However, it does not make sense to write an action of this form for all Δ except singletons. So one should work with the following action (with understanding $m^2 \to 0$ for AdS_3):

$$S = \int \mathrm{d}^{d+1}x\sqrt{g}\left(g^{\mu\nu}\partial_\mu A \partial_\nu B - m^2 AB + \frac{1}{2}B^2\right). \tag{14}$$

Given the form of singleton action in the bulk of AdS spacetime it remains to determine the pair of fields on the boundary (say C and D) which couples to the dipole fields A and B. Coupling of the fields C and D is of either $\int \mathrm{d}^{d-1}x(\alpha A_0 C + B_0 D)$ or $\int \mathrm{d}^{d-1}x(\alpha A_0 D + B_0 C)$.

Choosing the coupling $\int \mathrm{d}^d x(\alpha A_0 D + B_0 C)$ of boundary operators C and D, we get the two-point functions of logarithmic operators C and D on the boundary [13, 14] via AdS/CFT correspondence,

$$\langle C(\vec{x})C(\vec{x'})\rangle = 0$$
$$\langle C(\vec{x})D(\vec{x'})\rangle = \frac{\Delta}{|\vec{x}-\vec{x'}|^{2\Delta}} \qquad (15)$$
$$\alpha^2\langle D(\vec{x})D(\vec{x'})\rangle = \frac{\Delta}{2\Delta-d}\frac{1}{|\vec{x}-\vec{x'}|^{2\Delta}}\left[-2\ln\left(\frac{|\vec{x}-\vec{x'}|^2}{\epsilon}\right)+\frac{1}{\Delta}\right].$$

Standard normalization of equation (10) gives the value of $\alpha = 1/(2\Delta - d)$.

As $(x,\bar{x})$ has the interpretation of boundary coordinates, we see that there are logarithmic operators in boundary CFT if they are present in the bulk theory.

Thus, beginning with singleton action in bulk of AdS spacetime, we get logarithmic correlation functions on the boundary via AdS/CFT correspondence.

4. Correlation functions beyond $j = -\frac{1}{2}$

As discussed earlier, normalizable wavefunctions in AdS_3 exist only for $j < 0$ and are square integrable only for $j < -\frac{1}{2}$. To define correlation functions beyond $j = -\frac{1}{2}$, one has to take dimension-dependent normalization of the fields into account as discussed below.

Consider properly normalized primary fields of the $SL(2,\mathbb{R})$ WZNW model

$$\phi_j(x,z) = \frac{2j+1}{\pi}[(\gamma-x)(\bar{\gamma}-\bar{x})\,e^{2\phi} + e^{-2\phi}]^{2j}. \qquad (16)$$

The two-point function is completely determined by global conformal invariance,

$$\langle\phi_{j_1}(x_1,z_1)\phi_{j_2}(x_2,z_2)\rangle = B(j_1)\delta(j_1-j_2)|x_{12}|^{4j_1}|z_{12}|^{-4\Delta_{j_1}}. \qquad (17)$$

There is one more δ-function term on the right-hand side, but it vanishes if we restrict ourselves to $j < -\frac{1}{2}$. Coefficient $B(j)$ is evaluated in [28] and is given by

$$B(j) = -\pi^{2j+1}\left(\frac{\Gamma(1-b^2)}{\Gamma(1+b^2)}\right)^{2j+1}\frac{2j+1}{\pi}\frac{\Gamma(1+b^2(2j+1))}{\Gamma(1-b^2(2j+1))} \qquad b^2 = \frac{1}{k-2}.$$

In the supergravity limit $k \to \infty$, the two-point function becomes

$$\langle\phi_j(x_1,z_1)\phi_j(x_2,z_2)\rangle = -\frac{2j+1}{\pi}|x_{12}|^{4j}. \qquad (18)$$

This can be interpreted as two-point function of operators on the boundary which couples to bulk field $\phi_j(x,z)$ and is well defined for all values of $j < -\frac{1}{2}$ and vanishes at $j = -\frac{1}{2}$.

Now let us consider two-point functions in boundary conformal field theory in the context of AdS_3/CFT_2 correspondence.

Consider the Euclidean action for a massive scalar field in AdS_{d+1},

$$S[\phi] = \frac{1}{2}\int_{AdS_{d+1}} d^{d+1}x\sqrt{g}[g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + m^2\phi^2]. \qquad (19)$$

Scalar field in the bulk of AdS spacetime has the boundary limit

$$\phi(z,\vec{x}) \to z^{d-\Delta}[\phi_0(\vec{x}) + o(z^2)] + z^{\Delta}[A(\vec{x}) + o(z^2)] \qquad (20)$$

where $\phi_0(\vec{x})$ acts as a source function which couples the background field in the bulk and $A(\vec{x})$ denotes the small fluctuation around this background, and we have denoted boundary points by a vector and z is the radial coordinate in AdS.

Δ is given by the roots of the equation

$$\Delta(\Delta - d) = m^2. \qquad (21)$$

$\phi(z,\vec{x})$ can be constructed from $\phi_0(\vec{x})$ via

$$\phi(z,\vec{x}) = \int \mathrm{d}^d x' \, K_\Delta(z,\vec{x},\vec{x}')\phi_0(\vec{x}') \tag{22}$$

where $K_\Delta(z,\vec{x},\vec{x}')$ is bulk to boundary propagator [29, 30],

$$K_\Delta(z,\vec{x},\vec{x}') = \pi^{-d/2}\frac{\Gamma(\Delta)}{\Gamma\left(\Delta-\frac{d}{2}\right)}\left(\frac{z}{z^2+(\vec{x}-\vec{x}')^2}\right)^\Delta . \tag{23}$$

The two-point function of operators $O_i(\vec{x})$ with conformal dimension Δ, which couples to the fields $\phi_i(z,\vec{x})$ at the boundary, is given by

$$\langle O(\vec{x_1})O(\vec{x_2})\rangle = \frac{1}{\pi^{d/2}}\frac{(2\Delta-d)\Gamma(\Delta)}{\Gamma(\Delta-d/2)}\frac{1}{|(x_1-x_2)|^{2\Delta}}. \tag{24}$$

For AdS_3, we have $d=2$ and $2j=-\Delta$. Equation (21) has two roots

$$\Delta_\pm = 1 \pm \sqrt{1+m^2}.$$

Breitenlohner–Freedman [31] stability bound $m^2 > -1$ for a massive scalar field in AdS_3 implies that $\Delta_+ > 1$ and $\Delta_- > 0$.

Two-point function of operators on the boundary can be defined for $\Delta > 1$ for all Δ_+ and is given by [32]

$$\langle O_\Delta(x_1)O_\Delta(x_2)\rangle = \frac{1}{\pi}\frac{(2\Delta-2)\Gamma(\Delta)}{\Gamma(\Delta-1)}\frac{1}{(x_{12})^\Delta}. \tag{25}$$

The factor $\frac{\Gamma(\Delta)}{\Gamma(\Delta-1)}$ comes from the normalization of the bulk to boundary propagator.

To compute two-point functions of operators on the boundary CFT which couple to fields beyond $j=-\frac{1}{2}$ barrier, we need to define two-point function for the branch $0<\Delta\leqslant 1$, $\Delta=0$ being the unitary bound, by dimension-dependent renormalization of the fields [33]:

$$\langle O_\Delta(x_1)O_\Delta(x_2)\rangle = \frac{2}{\pi}\frac{\Gamma(\Delta)}{(\Delta-1)\Gamma(\Delta-1)}\frac{1}{(x_{12})^\Delta} = \frac{2}{\pi}\frac{1}{(x_{12})^\Delta}. \tag{26}$$

This is correct that conformally invariant two-point function defined for all $\Delta > 0$.

The fact that two-point functions of boundary operators can be defined for $\Delta < 1$ is suggestive that two-point functions of $SL(2,\mathbb{R})$ primaries can also be defined in the range $0 \geqslant j \geqslant -\frac{1}{2}$ (note: $\Delta = -2j$) by defining $\phi_i(x,z) \to \frac{\phi_i(x,z)}{2j+1}$. All such values of j violate Breitenlohner–Freedman stability bound in AdS_3.

We have considered only real values of j as it gives real conformal dimensions of the operators in boundary CFT. Meaning of complex j belonging to continuous series is still unclear from the viewpoint of AdS/CFT correspondence. Thus, it is possible to define correlation functions for all values of $j < 0$, which is the correct unitarity bound from the point of view of CFT.

5. Summary and discussion

We have discussed the possible occurrence of logarithmic operators in AdS_3 string theory. Origin of logarithmic operators lie in the fact that some fields are in indecomposable representation of extended symmetry algebra.

It was noticed that for $j=-\frac{1}{2}$ representation there are no logarithmic singularities in the coordinate space, but they are present in isotopic space. It was shown in [18] that OPE of $\phi_{-\frac{1}{2}}$ with any other operator does not have logarithmic terms, and $\phi_{-\frac{1}{2}}$ is analogue of prelogarithmic operator as in the Liouville theory. If we include $\phi_{-\frac{1}{2}}$ in the spectrum, we also have to include the other logarithmic operators in the spectrum, which have logarithmic correlation functions.

The singleton (fields in indecomposable representation of conformal algebra) action in the bulk gives boundary two-point functions, which are logarithmic in nature. Thus singletons play a special role in AdS/CFT correspondence.

It was possible to define correlation functions of the operators using a dimension-dependent renormalization beyond $j = -\frac{1}{2}$ representation using AdS/CFT correspondence which are well behaved for all values of j up to $j = 0$. The representation $j = 0$ is special and corresponds to singletons.

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